ON KADISON-SCHWARZ TYPE QUANTUM QUADRATIC OPERATORS ON $\mathbb{M}_2(\mathbb{C})$

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Abstract. In the present paper we study description of Kadison-Schwarz type quantum quadratic operators acting from $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$. Note that such kind of operator is a generalization of quantum convolution. By means of such a description we provide an example of q.q.o. which is not a Kadision-Schwartz operator. Moreover, we study dynamics of an associated nonlinear (i.e. quadratic) operators acting on the state space of $\mathbb{M}_2(\mathbb{C})$.

Mathematics Subject Classification: 46L35, 46L55, 46A37. 60J99.
Key words: quantum quadratic operators; Kadison-Schwarz operator.

1. Introduction

It is know that one of the main problems of quantum information is characterization of positive and completely positive maps on C^*-algebras. There are many papers devoted to this problem (see for example [5, 12, 22, 23]). In the literature the completely positive maps have proved to be of great importance in the structure theory of C*-algebras. However, general positive (order-preserving) linear maps are very intractable[12, ?]. It is therefore of interest to study conditions stronger than positivity, but weaker than complete positivity. Such a condition is called *Kadison-Schwarz property*, i.e a map ϕ satisfies the Kadison-Schwarz property if $\phi(a)^*\phi(a) \leq \phi(a^*a)$ holds for every a. Note that every unital completely positive map satisfies this inequality, and a famous result of Kadison states that any positive unital map satisfies the inequality for self-adjoint elements a. In [21] relations between n-positivity of a map ϕ and the Kadison-Schwarz property of certain map is established. Certain relations between complete positivity, positivite and the Kadison-Schwarz property have been considered in [1],[3],[2]. Some spectral and ergodic properties of Kadison-Schwarz maps were investigated in [8, 9, 20].

In [18] we have studied quantum quadratic operators (q.q.o.), i.e. maps from $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$, with the Kadison-Schwarz property. It is found some necessary conditions for the trace-preserving quadratic operators to be the Kadison-Schwarz ones. Since trace-preserving maps arise naturally in quantum information theory (see e.g. [19]) and other situations in which one wishes to restrict attention to a quantum system that should properly be considered a subsystem of a larger system with which it interacts. Note that in [6, 7] quantum quadratic operators acting on a von Neumann algebra were defined and studied. Certain ergodic properties of such operators were studied in [15, 16]. In the present paper we continue our investigation, i.e. we are going to study further properties of q.q.o. with Kadison-Schwarz property. We will provide an example of q.q.o. which is not a Kadision-Schwarz operator, and study its dynamics. We should stress that q.q.o. is a generalization of quantum convolution (see [24]). Some dynamical properties of quantum convolutions were investigated in [10].

Note that a description of bistochastic Kadison-Schwarz mappings from $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C})$ has been provided in [17].

2. Preliminaries

In what follows, by $\mathbb{M}_2(\mathbb{C})$ we denote an algebra of 2×2 matrices over complex filed $\mathbb{C}$. By $\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$ we mean tensor product of $\mathbb{M}_2(\mathbb{C})$ into itself. We note that such a product can be considered as an algebra of 4×4 matrices $\mathbb{M}_4(\mathbb{C})$ over $\mathbb{C}$. In the sequel $\mathbb{1}$ means an identity matrix, i.e. $\mathbb{1}=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$. By $S(\mathbb{M}_2(\mathbb{C}))$ we denote the set of all states (i.e. linear positive functionals which take value 1 at $\mathbb{1}$) defined on $\mathbb{M}_2(\mathbb{C})$.

Definition 2.1. A linear operator $\Delta:\mathbb{M}_2(\mathbb{C})\to\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$ is said to be

(a) – a *quantum quadratic operator (q.q.o.)* if it satisfies the following conditions:

(i) unital, i.e. $\Delta\mathbb{1}=\mathbb{1}\otimes\mathbb{1}$;

(ii) Δ is positive, i.e. $\Delta x\geq 0$ whenever $x\geq 0$;

(b) – a *Kadison-Schwarz operator (KS)* if it satisfies

$$\Delta(x^*x)\geq\Delta(x^*)\Delta(x)\ \ \text{for all } x\in\mathbb{M}_2(\mathbb{C}). \tag{2.1}$$

One can see that if Δ is unital and KS operator, then it is a q.q.o. A state $h\in S(\mathbb{M}_2(\mathbb{C}))$ is called *a Haar state* for a q.q.o. Δ if for every $x\in\mathbb{M}_2(\mathbb{C})$ one has

$$(h\otimes id)\circ\Delta(x)=(id\otimes h)\circ\Delta(x)=h(x)\mathbb{1}. \tag{2.2}$$

Remark 2.2. Let $U:\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})\to\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$ be a linear operator such that $U(x\otimes y)=y\otimes x$ for all $x,y\in\mathbb{M}_2(\mathbb{C})$. If a q.q.o. Δ satisfies $U\Delta=\Delta$, then Δ is called a *quantum quadratic stochastic operator*. Such a kind of operators were studied and investigated in [15].

Each q.q.o. Δ defines a conjugate operator $\Delta^*:(\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C}))^*\to\mathbb{M}_2(\mathbb{C})^*$ by

$$\Delta^*(f)(x)=f(\Delta x),\quad f\in(\mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C}))^*,\ x\in\mathbb{M}_2(\mathbb{C}). \tag{2.3}$$

One can define an operator V_Δ by

$$V_\Delta(\varphi)=\Delta^*(\varphi\otimes\varphi),\quad \varphi\in S(\mathbb{M}_2(\mathbb{C})), \tag{2.4}$$

which is called a *quadratic operator (q.c.)*. Thanks to the conditions (i),(ii) of Def. 2.1 the operator V_Δ maps $S(\mathbb{M}_2(\mathbb{C}))$ to $S(\mathbb{M}_2(\mathbb{C}))$.

3. Quantum quadratic operators with Kadison-Schwarz property on $\mathbb{M}_2(\mathbb{C})$

In this section we are going to describe quantum quadratic operators on $\mathbb{M}_2(\mathbb{C})$ as well as find necessary conditions for such operators to satisfy the Kadison-Schwarz property.

Recall [4] that the identity and Pauli matrices $\{\mathbb{1},\sigma_1,\sigma_2,\sigma_3\}$ form a basis for $\mathbb{M}_2(\mathbb{C})$, where

$$\sigma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\ \sigma_2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}\ \sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}.$$

In this basis every matrix $x\in\mathbb{M}_2(\mathbb{C})$ can be written as $x=w_0\mathbb{1}+\mathbf{w}\sigma$ with $w_0\in\mathbb{C}$, $\mathbf{w}=(w_1,w_2,w_3)\in\mathbb{C}^3$, here $\mathbf{w}\sigma=w_1\sigma_1+w_2\sigma_2+w_3\sigma_3$.

Lemma 3.1. [22] *The following assertions hold true:*

(a) x *is self-adjoint iff* $w_0,\mathbf{w}$ *are reals;*

(b) $\mathrm{Tr}(x)=1$ *iff* $w_0=0.5$*, here* Tr *is the trace of a matrix* x*;*

(c) $x>0$ *iff* $\|\mathbf{w}\|\leq w_0$*, where* $\|\mathbf{w}\|=\sqrt{|w_1|^2+|w_2|^2+|w_3|^2}$.

Note that any state $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$ can be represented by

$$\varphi(w_0\mathbb{1} + \mathbf{w}\sigma) = w_0 + \langle \mathbf{w}, \mathbf{f}\rangle, \tag{3.1}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ with $\|\mathbf{f}\| \leq 1$. Here as before $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{C}^3$. Therefore, in the sequel we will identify a state φ with a vector $\mathbf{f} \in \mathbb{R}^3$.

In what follows by τ we denote a normalized trace, i.e. $\tau(x) = \frac{1}{2}\operatorname{Tr}(x)$, $x \in \mathbb{M}_2(\mathbb{C})$,

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a q.q.o. with a Haar state τ. Then one has

$$\tau \otimes \tau(\Delta x) = \tau(\tau \otimes id)(\Delta(x)) = \tau(x)\tau(\mathbb{1}) = \tau(x), \quad x \in \mathbb{M}_2(\mathbb{C}),$$

which means that τ is an invariant state for Δ.

Let us write the operator Δ in terms of a basis in $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ formed by the Pauli matrices. Namely,

$$\begin{aligned} &\Delta\mathbb{1} = \mathbb{1} \otimes \mathbb{1}; \\ &\Delta(\sigma_i) = b_i(\mathbb{1} \otimes \mathbb{1}) + \sum_{j=1}^{3} b_{ji}^{(1)}(\mathbb{1} \otimes \sigma_j) + \sum_{j=1}^{3} b_{ji}^{(2)}(\sigma_j \otimes \mathbb{1}) + \sum_{m,l=1}^{3} b_{ml,i}(\sigma_m \otimes \sigma_l), \end{aligned}$$

where $i = 1, 2, 3$.

One can prove the following

Theorem 3.2. [18] *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a q.q.o. with a Haar state τ, then it has the following form:*

$$\Delta(x) = w_0\mathbb{1} \otimes \mathbb{1} + \sum_{m,l=1}^{3} \langle \mathbf{b}_{ml}, \overline{\mathbf{w}}\rangle \sigma_m \otimes \sigma_l, \tag{3.2}$$

where $x = w_0 + \mathbf{w}\sigma$, $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3})$.

Let us turn to the positivity of Δ. Given vector $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ put

$$\beta(\mathbf{f})_{ij} = \sum_{k=1}^{3} b_{ki,j} f_k. \tag{3.3}$$

Define a matrix $\mathbb{B}(\mathbf{f}) = (\beta(\mathbf{f})_{ij})_{ij=1}^{3}$.

Now given a state φ, (i.e. $\varphi(x) = \omega_0 + \langle \mathbf{w}, \mathbf{f}\rangle$, $\mathbf{f} \in \mathbb{R}^3$,$\|\mathbf{f}\| \leq 1$) by E_φ we denote the canonical conditional expectation defined by $E_\varphi(x \otimes y) = \varphi(x)y$, where $x, y \in \mathbb{M}_2(\mathbb{C})$.

By $\|\mathbb{B}(\mathbf{f})\|$ we denote a norm of the matrix $\mathbb{B}(\mathbf{f})$ associated with Euclidean norm in $\mathbb{C}^3$. Put

$$S = \{\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3 : \ p_1^2 + p_2^2 + p_3^2 \leq 1\}$$

and denote

$$|||\mathbb{B}||| = \sup_{\mathbf{f} \in S} \|\mathbb{B}(\mathbf{f})\|.$$

Proposition 3.3. *Let Δ be a q.q.o. with a Haar state τ, then $|||\mathbb{B}||| \leq 1$.*

Proof. From (3.2) we find

$$\begin{aligned} E_\varphi(\Delta(x)) &= w_0\mathbb{1} + \sum_{i,j=1}^{3} \langle \mathbf{b}_{i,j}, \overline{\mathbf{w}}\rangle f_i \sigma_j \\ &= w_0\mathbb{1} + \mathbb{B}(\mathbf{f})\mathbf{w}\sigma \end{aligned}$$

where $\varphi(x) = w_0 + \langle \mathbf{f}, \mathbf{w} \rangle$, $\mathbf{f} = (f_1, f_2, f_3) \in S$, and we have used $\varphi(\sigma_i) = f_i$ and

$$\sum_{i=1}^{3} \langle \mathbf{b}_{i,j}, \overline{\mathbf{w}} \rangle f_i = (\mathbb{B}(\mathbf{f})\mathbf{w})_j$$

Positivity of x yields that $E_\varphi(\Delta(x))$ is positive, for all states φ, since E_φ is a conditional expectation. Hence, according to Lemma 3.1 positivity of $E_\varphi(\Delta(x))$ equivalent to $\|\mathbb{B}(\mathbf{f})\mathbf{w}\| \leq w_0$ for all $\mathbf{f}$ and $\mathbf{w}$ with $\|\mathbf{w}\| < w_0$. Consequently, one finds that $\|\mathbb{B}(\mathbf{f})\| = \sup_{\|\mathbf{w}\| \leq 1} \|\mathbb{B}(\mathbf{f})\mathbf{w}\| \leq 1$, which yields the assertion. □

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a liner operator with a Haar state τ. Then due to Theorem 3.2 Δ has a form (3.2). Take arbitrary states $\varphi, \psi \in S(\mathbb{M}_2(\mathbb{C}))$ and $\mathbf{f}, \mathbf{p} \in S$ be the corresponding vectors (see (3.1)). Then one finds that

$$\Delta^*(\varphi \otimes \psi)(\sigma_k) = \sum_{i,j=1}^{3} b_{ij,k} f_i p_j, \quad k = 1, 2, 3.$$

Thanks to Lemma 3.1 the functional $\Delta^*(\varphi \otimes \psi)$ is a state if and only if the vector

$$\mathbf{f}_{\Delta^*(\varphi,\psi)} = \left(\sum_{i,j=1}^{3} b_{ij,1} f_i p_j, \sum_{i,j=1}^{3} b_{ij,2} f_i p_j, \sum_{i,j=1}^{3} b_{ij,3} f_i p_j \right).$$

satisfies $\|\mathbf{f}_{\Delta^*(\varphi,\psi)}\| \leq 1$.

So, we have the following

Proposition 3.4. *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a liner operator with a Haar state τ. Then $\Delta^*(\cdot \otimes \cdot)$ bilinear form is positive if and only if one holds*

$$\sum_{k=1}^{3} \left| \sum_{i,j=1}^{3} b_{ij,k} f_i p_j \right|^2 \leq 1 \quad \text{for all} \quad \mathbf{f}, \mathbf{p} \in S. \tag{3.4}$$

From the proof of Proposition 3.3 and the last proposition we get

Corollary 3.5. *Let $\mathbb{B}(\mathbf{f})$ be the corresponding matrix to an operator given by* (3.2). *Then $\|\|\mathbb{B}\|\| \leq 1$ if and only if* (3.4) *is satisfied.*

Remark 3.6. Note that characterizations of positive maps defined on $\mathbb{M}_2(\mathbb{C})$ were considered in [13] (see also [11]). Characterization of completely positive mappings from $\mathbb{M}_2(\mathbb{C})$ into itself with invariant state τ was established in [22] (see also [14]).

Next we would like to find some conditions for q.q.o. to be the Kadison-Schwarz ones.

Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a linear operator with a Haar state τ, then it has a form (3.2). Now we are going to find some conditions to the coefficients $\{b_{ml,k}\}$ when Δ is a Kadison-Schwarz operator. Given $x = w_0 + \mathbf{w}\sigma$ and state $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$ let us denote

$$\mathbf{x}_m = \big(\langle \mathbf{b}_{m1}, \mathbf{w} \rangle, \langle \mathbf{b}_{m2}, \mathbf{w} \rangle, \langle \mathbf{b}_{m3}, \mathbf{w} \rangle \big), \quad f_m = \varphi(\sigma_m), \tag{3.5}$$

$$\alpha_{ml} = \langle \mathbf{x}_m, \mathbf{x}_l \rangle - \langle \mathbf{x}_l, \mathbf{x}_m \rangle, \quad \gamma_{ml} = [\mathbf{x}_m, \overline{\mathbf{x}}_l] + [\overline{\mathbf{x}}_m, \mathbf{x}_l], \tag{3.6}$$

where $m, l = 1, 2, 3$. Note that here the numbers α_{ml} are skew-symmetric, i.e. $\overline{\alpha_{ml}} = -\alpha_{ml}$. By π we shall denote mapping $\{1, 2, 3, 4\}$ to $\{1, 2, 3\}$ defined by $\pi(1) = 2, \pi(2) = 3, \pi(3) = 1, \pi(4) = \pi(1)$.

Denote

$$\mathbf{q}(\mathbf{f}, \mathbf{w}) = \big(\langle \beta(\mathbf{f})_1, [\mathbf{w}, \overline{\mathbf{w}}] \rangle, \langle \beta(\mathbf{f})_2, [\mathbf{w}, \overline{\mathbf{w}}] \rangle, \langle \beta(\mathbf{f})_3, [\mathbf{w}, \overline{\mathbf{w}}] \rangle \big), \tag{3.7}$$

where $\beta(\mathbf{f})_m = \big(\beta(\mathbf{f})_{m1}, \beta(\mathbf{f})_{m2}, \beta(\mathbf{f})_{m3}\big)$ (see (3.3))

Theorem 3.7. [18] *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a Kadison-Schwarz operator with a Haar state τ, then it has the form* (3.2) *and the coefficients $\{b_{ml,k}\}$ satisfy the following conditions*

$$\|\mathbf{w}\|^2 \le i\sum_{m=1}^{3} f_m \alpha_{\pi(m),\pi(m+1)} + \sum_{m=1}^{3}\|\mathbf{x}_m\|^2 \tag{3.8}$$

$$\begin{aligned}\Big\|\mathbf{q}(\mathbf{f},\mathbf{w}) - i\sum_{m=1}^{3} f_m \gamma_{\pi(m),\pi(m+1)} - [\mathbf{x}_m, \overline{\mathbf{x}}_m]\Big\| \ \le\ & \|\mathbf{w}\|^2 - i\sum_{k=1}^{3} f_k \alpha_{\pi(k),\pi(k+1)} \\ & -\sum_{m=1}^{3}\|\mathbf{x}_m\|^2.\end{aligned} \tag{3.9}$$

for all $\mathbf{f} \in S, \mathbf{w} \in \mathbb{C}^3$. Here as before $\mathbf{x}_m = \big(\langle \mathbf{b}_{m1}, \mathbf{w}\rangle, \langle \mathbf{b}_{m2}, \mathbf{w}\rangle, \langle \mathbf{b}_{m3}, \mathbf{w}\rangle\big)$, $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3})$ and $\mathbf{q}(\mathbf{f},\mathbf{w})$, α_{ml} and γ_{ml} are defined in (3.7),(3.5),(3.6)*, respectively.*

Remark 3.8. The provided characterization with [12, 22] allows us to construct examples of positive or Kadison-Schwarz operators which are not completely positive (see next section).

Now we are going to give a general characterization of KS-operators. Let us first give some notations. For a given mapping $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$, by $\Delta(\sigma)$ we denote the vector $(\Delta(\sigma_1), \Delta(\sigma_2), \Delta(\sigma_3))$, and by $\mathbf{w}\Delta(\sigma)$ we mean the following

$$\mathbf{w}\Delta(\sigma) = w_1\Delta(\sigma_1) + w_2\Delta(\sigma_2) + w_3\Delta(\sigma_3), \tag{3.10}$$

where $\mathbf{w} \in \mathbb{C}^3$. Note that the last equality (3.10), due to the linearity of Δ, also can be written as $\mathbf{w}\Delta(\sigma) = \Delta(\mathbf{w}\sigma)$.

Theorem 3.9. *Let $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ be a unital $*$-preserving linear mapping. Then Δ is a KS-operator if and only if one has*

$$i[\mathbf{w}, \overline{\mathbf{w}}]\Delta(\sigma) + (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)) \le \mathbb{1} \otimes \mathbb{1}, \tag{3.11}$$

for all $\mathbf{w} \in \mathbb{C}^3$ with $\|\mathbf{w}\| = 1$.

Proof. Let $x \in M_2(\mathbb{C})$ be an arbitrary element, i.e. $x = w_0\mathbb{1} + \mathbf{w}\sigma$. Then $x^* = \overline{w_0}\mathbb{1} + \overline{\mathbf{w}}\sigma$. Therefore

$$x^*x = \big(|w_0|^2 + \|\mathbf{w}\|^2\big)\mathbb{1} + \big(w_0\overline{\mathbf{w}} + \overline{w_0}\mathbf{w} - i\big[\mathbf{w}, \overline{\mathbf{w}}\big]\big)\sigma$$

Consequently, we have

$$\Delta(x) = w_0\mathbb{1} \otimes \mathbb{1} + \mathbf{w}\Delta(\sigma), \quad \Delta(x^*) = \overline{w_0}\mathbb{1} \otimes \mathbb{1} + \overline{\mathbf{w}}\Delta(\sigma) \tag{3.12}$$

$$\Delta(x^*x) = \big(|w_0|^2 + \|\mathbf{w}\|^2\big)\mathbb{1} \otimes \mathbb{1} + \big(w_0\overline{\mathbf{w}} + \overline{w_0}\mathbf{w} - i\big[\mathbf{w}, \overline{\mathbf{w}}\big]\big)\Delta(\sigma) \tag{3.13}$$

$$\Delta(x)^*\Delta(x) = |w_0|^2\mathbb{1} \otimes \mathbb{1} + \big(w_0\overline{\mathbf{w}} + \overline{w_0}\mathbf{w}\big)\Delta(\sigma) + (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)) \tag{3.14}$$

From (3.13)-(3.14) one gets

$$\Delta(x^*x) - \Delta(x)^*\Delta(x) = \|\mathbf{w}\|^2\mathbb{1} \otimes \mathbb{1} - i\big[\mathbf{w}, \overline{\mathbf{w}}\big]\Delta(\sigma) - (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)).$$

So, the positivity of the last equality implies that

$$\|\mathbf{w}\|^2\mathbb{1} \otimes \mathbb{1} - i\big[\mathbf{w}, \overline{\mathbf{w}}\big]\Delta(\sigma) - (\mathbf{w}\Delta(\sigma))(\overline{\mathbf{w}}\Delta(\sigma)) \ge 0.$$

Now dividing both sides by $\|\mathbf{w}\|^2$ we get the required inequality. Hence, this completes the proof. $\square$

4. An Example of q.q.o. which is not Kadision-Schwarz one

In this section we are going to study dynamics of (5.2) for a special class of quadratic operators. Such a class operators associated with the following matrix $\{b_{ij,k}\}$ given by:

$$\begin{array}{lll} b_{11,1}=\varepsilon; & b_{11,2}=0; & b_{11,3}=0; \\ b_{12,1}=0; & b_{12,2}=0; & b_{12,3}=\varepsilon; \\ b_{13,1}=0; & b_{13,2}=\varepsilon; & b_{13,3}=0; \\ b_{22,1}=0; & b_{22,2}=\varepsilon; & b_{22,3}=0; \\ b_{23,1}=\varepsilon; & b_{23,2}=0; & b_{23,3}=0; \\ b_{33,1}=0; & b_{33,2}=0; & b_{33,3}=\varepsilon; \end{array}$$

and $b_{ij,k}=b_{ji,k}$.

Via (3.2) we define a liner operator Δ_ε, for which τ is a Haar state. In the sequel we would like to find some conditions to ε which ensures positivity of Δ_ε.

It is easy that for given $\{b_{ijk}\}$ one can find a form of Δ_ε as follows

$$\begin{aligned} \Delta_\varepsilon(x) \quad = \quad & w_0\mathbb{1}\otimes\mathbb{1}+\varepsilon\omega_1\sigma_1\otimes\sigma_1+\varepsilon\omega_3\sigma_1\otimes\sigma_2+\varepsilon\omega_2\sigma_1\otimes\sigma_3 \\ & +\varepsilon\omega_3\sigma_2\otimes\sigma_1+\varepsilon\omega_2\sigma_2\otimes\sigma_2+\varepsilon\omega_1\sigma_2\otimes\sigma_3 \\ & +\varepsilon\omega_2\sigma_3\otimes\sigma_1+\varepsilon\omega_1\sigma_3\otimes\sigma_2+\varepsilon\omega_3\sigma_3\otimes\sigma_3, \end{aligned} \tag{4.1}$$

where as before $x=w_0\mathbb{1}+\mathbf{w}\sigma$.

Let us first establish satisfaction the condition (3.4) under some constrains to ε.

Lemma 4.1. *Let $|\varepsilon|\leq\frac{1}{\sqrt{3}}$ be satisfied, then (3.4) holds.*

Proof. Take any $\mathbf{f}=(f_1,f_2,f_3),\mathbf{p}=(p_1,p_2,p_3)\in S$. Then one finds

$$\begin{aligned} \sum_{k=1}^{3}\left|\sum_{i,j=1}^{3}b_{ij,k}f_ip_j\right|^2 \quad = \quad & \varepsilon^2(|f_1p_1+f_3p_2+f_2p_3|^2+|f_3p_1+f_2p_2+f_1p_3|^2 \\ & +|f_2p_1+f_1p_2+f_3p_3|^2) \\ \leq \quad & \varepsilon^2((f_1^2+f_2^2+f_3^2)(p_1^2+p_2^2+p_3^2)+(f_3^2+f_2^2+f_1^2)(p_1^2+p_2^2+p_3^2) \\ & +(p_1^2+p_2^2+p_3^2)(f_2^2+f_1^2+f_3^2)) \\ \leq \quad & \varepsilon^2(1+1+1)=3\varepsilon^2\leq 1. \end{aligned}$$

□

Now let us turn to the positivity of (4.1).

Theorem 4.2. *A linear operator Δ_ε is a q.q.o. if and only if $|\varepsilon|\leq\frac{1}{3}$.*

Proof. Let $x=w_0\mathbb{1}+\mathbf{w}\sigma$ be a positive element from $\mathbb{M}_2(\mathbb{C})$. Let us show positivity of the matrix $\Delta_\varepsilon(x)$. To do it, we rewrite (4.1) as follows $\Delta_\varepsilon(x)=w_0\mathbb{1}+\varepsilon\mathbf{B}$, here

$$\mathbf{B}=\begin{pmatrix} \omega_3 & \omega_2-i\omega_1 & \omega_2-i\omega_1 & \omega_1-2i\omega_3-\omega_2 \\ \omega_2+i\omega_1 & -\omega_3 & \omega_1+\omega_2 & -\omega_2+i\omega_1 \\ \omega_2+i\omega_1 & \omega_1+\omega_2 & -\omega_3 & -\omega_2+i\omega_1 \\ \omega_1+2i\omega_3-\omega_2 & -\omega_2-i\omega_1 & -\omega_2-i\omega_1 & \omega_3 \end{pmatrix},$$

where positivity of x yields that $w_0,\omega_1,\omega_2,\omega_3$ are real numbers. In what follows, without loss of generality, we may assume that $w_0=1$, and therefore $\|\mathbf{w}\|\leq 1$. It is known that positivity of $\Delta_\varepsilon(x)$ is equivalent to positivity of the eigenvalues of $\Delta_\varepsilon(x)$.

Let us first examine eigenvalues of $\mathbf{B}$. Simple algebra shows us that all eigenvalues of $\mathbf{B}$ can be written as follows

$$\lambda_1(\mathbf{w}) = \omega_1 + \omega_2 + \omega_3 + 2\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3}$$

$$\lambda_2(\mathbf{w}) = \omega_1 + \omega_2 + \omega_3 - 2\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3}$$

$$\lambda_3(\mathbf{w}) = \lambda_4(\mathbf{w}) = -\omega_1 - \omega_2 - \omega_3$$

Now examine maximum of the functions $\lambda_1(\mathbf{w}), \lambda_2(\mathbf{w}), \lambda_3(\mathbf{w}), \lambda_4(\mathbf{w})$ on the ball $\|\mathbf{w}\| \leq 1$. One can see that

$$|\lambda_3(\mathbf{w})| = |\lambda_4(\mathbf{w})| \leq \sum_{k=1}^{3} |\omega_k| \leq \sqrt{3}\sum_{k=1}^{3} |\omega_k|^2 \leq \sqrt{3}$$

Now let us rewrite $\lambda_1(\mathbf{w})$ and $\lambda_2(\mathbf{w})$ as follows

$$\lambda_1(\mathbf{w}) = \omega_1 + \omega_2 + \omega_3 + \frac{2}{\sqrt{2}}\sqrt{3(\omega_1^2 + \omega_2^2 + \omega_3^2) - (\omega_1 + \omega_2 + \omega_3)^2} \tag{4.2}$$

$$\lambda_2(\mathbf{w}) = \omega_1 + \omega_2 + \omega_3 - \frac{2}{\sqrt{2}}\sqrt{3(\omega_1^2 + \omega_2^2 + \omega_3^2) - (\omega_1 + \omega_2 + \omega_3)^2} \tag{4.3}$$

One can see that

$$\lambda_k(h\omega_1, h\omega_2, h\omega_3) = h\lambda_k(\omega_1, \omega_2, w_3)$$

for any $h \in \mathbb{R}$ $(k = 1, 2)$. Therefore, the functions $\lambda_k(\mathbf{w})$, $k = 1, 2$ reach their maximum on the sphere $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$ (i.e. $\|\mathbf{w}\| = 1$). Hence, denoting $t = \omega_1 + \omega_2 + \omega_3$ from (4.2) and (4.3) we introduce the following functions

$$g_1(t) = t + \frac{2}{\sqrt{2}}\sqrt{3 - t^2}, \qquad g_2(t) = t - \frac{2}{\sqrt{2}}\sqrt{3 - t^2}$$

where $|t| \leq \sqrt{3}$.

One can find that the critical values of g_1 are $t = \pm 1$, and the critical value of g_2 is $t = -1$. Consequently, the maximum of g_1 and g_2 on $|t| \leq \sqrt{3}$ are the following:

$$\max_{|t|\leq\sqrt{3}} |g_1(t)| = 3, \quad \max_{|t|\leq\sqrt{3}} |g_2(t)| = 3;$$

Therefore, we conclude that

$$\max_{|\mathbf{w}|\leq 1} |\lambda_1(\mathbf{w})| = 3, \quad \max_{|\mathbf{w}|\leq 1} |\lambda_2(\mathbf{w})| = 3;$$

It is known that for the spectrum of $\mathbb{1} + \varepsilon\mathbf{B}$ one has

$$Sp(\mathbb{1} + \varepsilon\mathbf{B}) = 1 + \varepsilon Sp(\mathbf{B})$$

Therefore,

$$Sp(\mathbb{1} + \varepsilon\mathbf{B}) = \{1 + \varepsilon\lambda_k(\mathbf{w}) : k = \overline{1, 4}\}$$

So, if

$$|\varepsilon| \leq \frac{1}{\max\limits_{\|\mathbf{w}\|\leq 1} |\lambda_k(\mathbf{w})|}, \quad k = \overline{1, 4}$$

Then one can see $1+\varepsilon\lambda_k(\mathbf{w})\geq 0$ for all $\|\mathbf{w}\|\leq 1$, $\quad k=\overline{1,4}$. This implies that the matrix $\mathbb{I}+\varepsilon\mathbf{B}$ is positive for all $\mathbf{w}$ with $\|\mathbf{w}\|\leq 1$. This yields the required assertion. $\square$

From Lemma 4.1 and Theorem 4.2 we conclude that if $\varepsilon\in(\frac{1}{3},\frac{1}{\sqrt{3}}]$ then the operator Δ_ε is not positive, while (3.4) is satisfied.

Theorem 4.3. *Let $\varepsilon=\frac{1}{3}$ then corresponding q.q.o. Δ_ε is not KS-operator.*

Proof. It is enough to show dissatisfaction the inequality (3.9) at some value $\mathbf{w}:\ \|\mathbf{w}\|\leq 1$ and $\mathbf{f}=(f_1,f_1,f_2)$.

Assume that $\mathbf{f}=(1,0,0)$, then a little algebra shows that (3.9) reduces to the following one

$$\sqrt{A+B+C}\leq D \tag{4.4}$$

where

$$\begin{aligned}
A &= |\varepsilon(\overline{\omega}_2\omega_3-\overline{\omega}_3\omega_2)-i\varepsilon^2(2\overline{\omega}_2\omega_3-2|\omega_1|^2-\overline{\omega}_2\omega_1+\overline{\omega}_1\omega_2-\overline{\omega}_1\omega_3+\overline{\omega}_3\omega_1)|^2\\
B &= |\varepsilon(\overline{\omega}_1\omega_2-\overline{\omega}_2\omega_1)-i\varepsilon^2(2\overline{\omega}_1\omega_2-2|\omega_3|^2-\overline{\omega}_1\omega_3+\overline{\omega}_3\omega_1-\overline{\omega}_3\omega_2+\overline{\omega}_2\omega_3)|^2\\
C &= |\varepsilon(\overline{\omega}_3\omega_1-\overline{\omega}_1\omega_3)-i\varepsilon^2(2\overline{\omega}_3\omega_1-2|\omega_2|^2-\overline{\omega}_3\omega_2+\overline{\omega}_2\omega_3-\overline{\omega}_2\omega_1+\overline{\omega}_1\omega_2)|^2\\
D &= (1-3|\varepsilon|^2)(|\omega_1|^2+|\omega_2|^2+|\omega_3|^2)\\
&\quad -i\varepsilon^2(\overline{\omega}_3w_2-\overline{\omega}_2\omega_3+\overline{\omega}_2\omega_1-\overline{\omega}_1\omega_2+\overline{\omega}_1\omega_3-\overline{\omega}_3\omega_1)
\end{aligned}$$

Now choose $\mathbf{w}$ as follows:

$$\omega_1=-\frac{1}{9};\qquad \omega_2=\frac{5}{36};\qquad \omega_3=\frac{5i}{27}$$

Then calculations show that

$$\begin{aligned}
A &= \frac{9594}{19131876};\qquad B=\frac{19625}{86093442};\\
C &= \frac{1625}{3779136};\qquad D=\frac{589}{17496}.
\end{aligned}$$

Hence, we find

$$\sqrt{\frac{9594}{19131876}+\frac{19625}{86093442}+\frac{1625}{3779136}}>\frac{589}{17496}$$

which means that (4.4) is not satisfied. Hence, Δ_ε is not a KS-operator at $\varepsilon=1/3$. $\square$

Recall that a linear operator $T:\mathbb{M}_k(\mathbb{C})\to\mathbb{M}_m(\mathbb{C})$ is *completely positive* if for any positive matrix $\big(a_{ij})\big)_{i,j=1}^n\in\mathbb{M}_k(\mathbb{M}_n(\mathbb{C}))$ the matrix $\big(T(a_{ij})\big)_{i,j=1}^n$ is positive for all $n\in\mathbb{N}$. Now we are interested when the operator Δ_ε is completely positive. It is know [5] that the complete positivity of Δ_ε is equivalent to the positivity of the following matrix

$$\hat{\Delta}_\varepsilon=\begin{pmatrix}\Delta_\varepsilon(e_{11}) & \Delta_\varepsilon(e_{12})\\ \Delta_\varepsilon(e_{21}) & \Delta_\varepsilon(e_{22})\end{pmatrix}$$

here e_{ij} are the matrix units in $\mathbb{M}_2(\mathbb{C})$.

From (4.1) one can calculate that

$$\Delta_\varepsilon(e_{11}) = \frac{1}{2}\mathbb{1}\otimes\mathbb{1} + \varepsilon B_{11}, \quad \Delta_\varepsilon(e_{22}) = \frac{1}{2}\mathbb{1}\psi\mathbb{1} - \varepsilon B_{11}$$

$$\Delta_\varepsilon(e_{12}) = \varepsilon B_{12}, \quad \Delta_\varepsilon(e_{21}) = \varepsilon B_{12}^*$$

where

$$B_{11} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -i \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ i & 0 & 0 & \frac{1}{2} \end{pmatrix}, \quad B_{12} = \begin{pmatrix} 0 & 0 & 0 & \frac{1-i}{2} \\ i & 0 & \frac{1+i}{2} & 0 \\ i & \frac{1+i}{2} & 0 & 0 \\ \frac{1-i}{2} & -i & -i & 0 \end{pmatrix}$$

Hence, we find

$$2\hat{\Delta}_\varepsilon = \mathbb{1}_8 + \epsilon\mathbb{B}$$

where $\mathbb{1}_8$ is the unit matrix in $\mathbb{M}_8(\mathbb{C})$ and

$$\mathbb{B} = \begin{pmatrix} 1 & 0 & 0 & -2i & 0 & 0 & 0 & 1-i \\ 0 & -1 & 0 & 0 & 2i & 0 & 1+i & 0 \\ 0 & 0 & -1 & 0 & 2i & 1+i & 0 & 0 \\ 2i & 0 & 0 & 1 & 1-i & -2i & -2i & 0 \\ 0 & -2i & -2i & 1+i & -1 & 0 & 0 & 2i \\ 0 & 0 & 1-i & 2i & 0 & 1 & 0 & 0 \\ 0 & 1-i & 0 & 2i & 0 & 0 & 1 & 0 \\ 1+i & 0 & 0 & 0 & -2i & 0 & 0 & -1 \end{pmatrix}$$

So, the matrix $\hat{\Delta}_\varepsilon$ is positive if and only if

$$|\varepsilon| \leq \frac{1}{\lambda_{\max}(\mathbb{B})},$$

where $\lambda_{\max}(\mathbb{B}) = \max\limits_{\lambda\in Sp(\mathbb{B})}|\lambda|$.

One can easily calculate that $\lambda_{\max}(\mathbb{B}) = 3\sqrt{3}$. Therefore, we have the following

Theorem 4.4. *Let $\Delta_\varepsilon : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})\otimes\mathbb{M}_2(\mathbb{C})$ be given by* (4.1). *Then Δ_ε is completely positive if and only if $|\varepsilon| \leq \frac{1}{3\sqrt{3}}$.*

5. Dynamics of Δ_ε

Let Δ be a q.q.o. on $\mathbb{M}_2(\mathbb{C})$. Let us consider the quadratic operator, which is defined as $V_\Delta(\varphi) = \Delta^*(\varphi\otimes\varphi)$, $\varphi \in S(\mathbb{M}_2(\mathbb{C}))$. From Theorem 3.2 and Corollary 3.5 one can see that the defined operator V_Δ maps $S(\mathbb{M}_2(\mathbb{C}))$ into itself if and only if $|||\mathbb{B}||| \leq 1$ or equivalently (3.4). From (3.2) we find that

$$V_\Delta(\varphi)(\sigma_k) = \sum_{i,j=1}^{3} b_{ij,k} f_i f_j, \quad \mathbf{f} \in S. \tag{5.1}$$

Here as before $S = \{\mathbf{f} = (f_1, f_2 f p_3) \in \mathbb{R}^3 : \ f_1^2 + f_2^2 + f_3^2 \leq 1\}$.

Remark 5.1. We have stress here that the condition (3.4) is a necessary condition for Δ to be a positive operator (see Lemma 4.1 and Theorem 4.2).

So, (5.1) suggests us to consider of the following nonlinear operator $V : S \to S$ defined by

$$V(\mathbf{f})_k = \sum_{i,j=1}^{3} b_{ij,k} f_i f_j, \quad k = 1, 2, 3. \tag{5.2}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in S$.

It is worth to mention that uniqueness of the fixed point (i.e. $(0,0,0)$) of the operator given by (5.2) has been investigated in [18].

In this section, we are going to study dynamics of the quadratic operator V_ε corresponding to Δ_ε (see (4.1)), which has the following form

$$\begin{cases} V_\varepsilon(f)_1 = \varepsilon(f_1^2 + 2f_2f_3) \\ V_\varepsilon(f)_2 = \varepsilon(f_2^2 + 2f_1f_3) \\ V_\varepsilon(f)_3 = \varepsilon(f_3^2 + 2f_1f_2) \end{cases}$$

Note that due to Proposition 3.4 and Lemma 4.1 the quadratic operator V_ε maps S into itself. Recall that a vector $\mathbf{f} \in S$ is a fixed point of V_ε if $V_\varepsilon(\mathbf{f}) = \mathbf{f}$. Clearly $(0,0,0)$ is a fixed point of V_ε. Let us find others. To do it, we need to solve the following equation

$$\begin{cases} \varepsilon(f_1^2 + 2f_2f_3) = f_1 \\ \varepsilon(f_2^2 + 2f_1f_3) = f_2 \\ \varepsilon(f_3^2 + 2f_1f_2) = f_3 \end{cases} \tag{5.3}$$

We have the following

Proposition 5.2. *If $|\varepsilon| < \frac{1}{\sqrt{3}}$ then V_ε has a unique fixed point $(0,0,0)$. If $|\varepsilon| = \frac{1}{\sqrt{3}}$ then V_ε has the following fixed points $(0,0,0)$ and $(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$.*

Proof. It is clear that $(0,0,0)$ is a fixed point of V_ε. If $f_k = 0$, for some $k \in \{1,2,3\}$ then due to $|\varepsilon| \le \frac{1}{\sqrt{3}}$, one can see that only solution of (5.3) belonging to S is $f_1 = f_2 = f_3 = 0$. Therefore, we assume that $f_k \neq 0$ $(k = 1,2,3)$. So, from (5.3) one finds

$$\begin{cases} \frac{{f_1}^2 + 2f_2f_3}{{f_2}^2 + 2f_1f_3} = \frac{f_1}{f_2} \\ \frac{{f_1}^2 + 2f_2f_3}{{f_3}^2 + 2f_1f_2} = \frac{f_1}{f_3} \\ \frac{{f_2}^2 + 2f_1f_3}{{f_3}^2 + 2f_1f_2} = \frac{f_2}{f_3} \end{cases} \tag{5.4}$$

Denoting

$$x = \frac{f_1}{f_2}, \ y = \frac{f_1}{f_3}, \ z = \frac{f_2}{f_3} \tag{5.5}$$

From (5.4) it follows that

$$\begin{cases} x\left(\frac{x\left(1+\frac{2}{xy}\right)}{1+\frac{2x}{z}} - 1\right) = 0 \\ y\left(\frac{y\left(1+\frac{2}{xy}\right)}{1+2yz} - 1\right) = 0 \\ z\left(\frac{z\left(1+\frac{2x}{z}\right)}{1+2yz} - 1\right) = 0 \end{cases} \tag{5.6}$$

According to our assumption x, y, z are nonzero, so from (5.6) one gets

$$\left\{\begin{array}{l} \frac{x\left(1+\frac{2}{xy}\right)}{1+\frac{2x}{z}}=1 \\ \frac{y\left(1+\frac{2}{xy}\right)}{1+2yz}=1 \\ \frac{z\left(1+\frac{2x}{z}\right)}{1+2yz}=1 \end{array}\right. \tag{5.7}$$

where $2x \neq -z$ and $2yz \neq -1$.

Dividing the second equality of (5.7) to the first one of (5.7) we find

$$\frac{y\left(1+\frac{2x}{z}\right)}{x(1+2yz)}=1$$

which with $xz = y$ yields

$$y + 2x^2 = x + 2y^2.$$

Simplifying the last equality one gets

$$(y - x)(1 - 2(y + x)) = 0.$$

This means that either $y = x$ or $x + y = \frac{1}{2}$.

Assume that $x = y$. Then from $xz = y$, one finds $z = 1$. Moreover, from the second equality of (5.7) we have $y + \frac{2}{y} = 1 + 2y$. So, $y^2 + y - 2 = 0$ therefore, the solutions of the last one are $y_1 = 1, y_2 = -2$. Hence, $x_1 = 1, x_2 = -2$.

Now suppose that $x + y = \frac{1}{2}$, then $x = \frac{1}{2} - y$. We note that $y \neq 1/2$, since $x \neq 0$. So, from the second equality of (5.7) we find

$$y + \frac{4}{1-2y} = 1 + \frac{4y^2}{1-2y}.$$

So, $2y^2 - y - 1 = 0$ which yields the solutions $y_3 = -\frac{1}{2}, y_4 = 1$. Therefore, we obtain $x_3 = 1$, $z_3 = -\frac{1}{2}$ and $x_4 = -\frac{1}{2}$, $z_4 = -2$.

Consequently, solutions of (5.7) are the following ones

$$(0,0,1),\ (1,1,1),\ \left(1,-\frac{1}{2},-\frac{1}{2}\right),\ \left(-\frac{1}{2},1,-2\right),\ (-2,-2,1).$$

Now owing to (5.5) and we need to solve the following equations

$$\left\{\begin{array}{l} \frac{f_1}{f_2}=x_k, \\ \frac{f_2}{f_3}=z_k, \end{array}\right. \qquad k=\overline{1,5}. \tag{5.8}$$

According to our assumption $f_k \neq 0$, therefore we consider cases when $x_k z_k \neq 0$.

Now let us start to consider several cases:

CASE 1. Let $x_2 = 1$, $z_2 = 1$. Then from (5.8) one gets $f_1 = f_2 = f_3$. So, from (5.3) we find $3\varepsilon {f_1}^2 = f_1$, i.e. $f_1 = \frac{1}{3\varepsilon}$. Now taking into account ${f_1}^2 + {f_2}^2 + {f_3}^2 \leq 1$ one gets $\frac{1}{3\varepsilon^2} \leq 1$. From the last inequality we have $|\varepsilon| \geq \frac{1}{\sqrt{3}}$. Due to Lemma 4.1 we obtain $|\varepsilon| = \frac{1}{\sqrt{3}}$. Hence, in this case a solution is $\left(\pm\frac{1}{\sqrt{3}}; \pm\frac{1}{\sqrt{3}}; \pm\frac{1}{\sqrt{3}}\right)$.

CASE 2. Let $x_2 = 1$, $z_2 = -1/2$. Then from (5.8) one finds $f_1 = f_2, 2f_2 = -f_3$. Substituting the last ones to (5.3) we get $f_1 + 3{f_1}^2\varepsilon = 0$. Then, we have $f_1 = -\frac{1}{3\varepsilon}, f_2 = -\frac{1}{3\varepsilon}, f_3 = \frac{2}{3\varepsilon}$. Taking into account ${f_1}^2 + {f_2}^2 + {f_3}^2 \leq 1$ we find $\frac{1}{9\varepsilon^2} + \frac{4}{9\varepsilon^2} + \frac{1}{9\varepsilon^2} \leq 1$. This means $|\varepsilon| \geq \sqrt{\frac{2}{3}}$, but it contradicts to Lemma 4.1. Hence, in this case there is not solution belonging to S.

Using the same argument for the rest cases we conclude the absence of solutions. This completes the proof. □

Now we are going to study dynamics of operator V_ε.

Theorem 5.3. *Let V_ε be a quadratic operator corresponding to (4.1). Then the following assertions hold true:*

(i) *if $|\varepsilon| < 1/\sqrt{3}$, then for any $\mathbf{f} \in S$ with $\mathbf{f} \neq (0,0,0)$ one has $V_\varepsilon^n(\mathbf{f}) \to (0,0,0)$ as $n \to \infty$.*

(ii) *if $|\varepsilon| = 1/\sqrt{3}$, then for any $\mathbf{f} \in S$ with $\mathbf{f} \notin \big\{(0,0,0), (\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})\big\}$ one has $V_\varepsilon^n(\mathbf{f}) \to (0,0,0)$ as $n \to \infty$.*

Proof. Let us consider the following function $\rho(\mathbf{f}) = f_1^2 + f_2^2 + f_3^2$. Then we have

$$\begin{aligned}
\rho(V(\mathbf{f})) &= \varepsilon^2\big((f_1^2+2f_2f_3)^2 + (f_2^2+2f_1f_3)^2 + (f_3^2+2f_1f_2)^2\big) \\
&\leq \varepsilon^2\big(f_1^2 + 2|f_2||f_3| + f_2^2 + 2|f_1||f_3| + f_3^2 + 2|f_1||f_2|\big) \\
&\leq \varepsilon^2\big(f_1^2+f_2^2+f_3^2+f_2^2+f_1^2+f_3^2+f_3^2+f_1^2+f_2^2\big) \\
&= 3\varepsilon^2(f_1^2+f_2^2+f_3^2) = 3\varepsilon^2\rho(\mathbf{f})
\end{aligned}$$

This means

$$\rho(V(\mathbf{f})) \leq 3\varepsilon^2\rho(\mathbf{f}). \tag{5.9}$$

Due to $\varepsilon^2 \leq \frac{1}{3}$ from (5.9) one finds

$$\rho(V^{n+1}(\mathbf{f})) \leq \rho(V^n(\mathbf{f})),$$

which yields that the sequence $\{\rho(V^n(\mathbf{f}))\}$ is convergent. Next we would like to find the limit of the sequence $\{\rho(V^n(\mathbf{f}))\}$.

(i). First assume that $|\varepsilon| < \frac{1}{3}$, then from (5.9) we obtain

$$\rho(V^n(\mathbf{f})) \leq 3\varepsilon^2\rho(V^{n-1}(\mathbf{f})) \leq \cdots \leq (3\varepsilon^2)^n\rho(\mathbf{f}).$$

This yields that $\rho(V^n(\mathbf{f})) \to 0$ as $n \to \infty$, for all $\mathbf{f} \in S$.

(ii). Now let $|\varepsilon| = \frac{1}{3}$. Then consider two distinct subcases.

Case (a). Let $f_1^2+f_2^2+f_3^2 < 1$ and denote $d = f_1^2+f_2^2+f_3^2$. Then one gets

$$\begin{aligned}
\rho(V(\mathbf{f})) &\leq \varepsilon^2\big((f_1^2+2|f_2||f_3|)^2 + (f_2^2+2|f_1||f_3|)^2 + (f_3^2+2|f_1||f_2|)^2\big) \\
&\leq \varepsilon^2\big((f_1^2+f_2^2+f_3^2)^2 + (f_2^2+f_1^2+f_3^2)^2 + (f_3^2+f_1^2+f_2^2)^2\big) \\
&= 3\varepsilon^2 d^2 = dd = d\rho(\mathbf{f}).
\end{aligned}$$

Hance, we have $\rho(V(\mathbf{f})) \leq d\rho(\mathbf{f})$. This means $\rho(V^n(\mathbf{f})) \leq d^n\rho(\mathbf{f}) \to 0$. Hence, $\rho(V^n(\mathbf{f})) \to 0$ as $n \to \infty$.

Case (b). Now take $f_1^2+f_2^2+f_3^2 = 1$ and assume that $\mathbf{f}$ is not a fixed point. Due to Lemma 5.2 this means that $f_i \neq f_j$ for $i \neq j$. Then we find

$$V_\varepsilon(f)_1 = \varepsilon(f_1^2+2f_2f_3) = \varepsilon(1-f_2^2-f_3^2+2f_2f_3) = \varepsilon(1-(f_2-f_3)^2) < \frac{1}{\sqrt{3}}.$$

Similarly one gets

$$V_\varepsilon(f)_2 = \varepsilon(1-(f_1-f_3)^2) < \frac{1}{\sqrt{3}},$$

$$V_\varepsilon(f)_3 = \varepsilon(1-(f_1-f_2)^2) < \frac{1}{\sqrt{3}}.$$

From $f_i \neq f_j$ ($i \neq j$) we conclude that $V(f)_1^2 + V(f)_2^2 + V(f)_3^2 < 1$, therefore according case (a), one finds that $\rho(V^n(\mathbf{f})) \to 0$ as $n \to \infty$.

$\square$

Acknowledgement

The author acknowledges the MOHE Grant FRGS11-022-0170. He also thanks the Junior Associate scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

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